Scaling hypothesis leading to extended self-similarity in turbulence

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A scaling hypothesis leading to extended self-similarity (ESS) for structure functions (the qth order moments of the *magnitude* of the longitudinal component velocity differences) in isotropic, homogeneous turbulence is proposed. This is done by generalizing the scale variable r to $rg(r/L)$, with a crossover function g. By extending the refined self-similarity, it is shown that the presented scaling also leads to ESS for structure functions of the energy dissipation rate fluctuations, and to ESS bridging relations between velocity and dissipation rate moments. Extended self-similarity on the basis of a universal crossover function g strictly holds toward the outer scale (L) range only. Yet we find at least approximate ESS toward the viscous, inner scale (l) range. Furthermore, the probability densities for the velocity differences and the energy dissipation rate fluctuations which are compatible with this ESS are offered.

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I. INTRODUCTION

Intermittency corrections to the K41 theory [1] of the velocity structure functions, or velocity difference moments of order q, $S_q^u(r)=\langle u_r^q\rangle$, u_r being the *magnitude* of the difference of the longitudinal velocity components between two points separated by the distance r in homogeneous isotropic turbulence, are one of the most challenging problems not only in hydrodynamics but also from the nonequilibrum statistical mechanical point of view [2–4].

If r is in the inertial subrange (ISR), the velocity structure functions scale with a power law

$$S_q^u(r)\propto r^{\zeta(q)}. \tag{1}$$

The scaling exponents $\zeta(q)$ are the important objects. In particular, one asks, about their dependence on q (see, e.g., Refs. [2,3]). The ISR is well defined only if the Reynolds number is quite large. $\zeta(q)$ therefore have mainly been studied in so-called fully developed turbulence. From experimental and numerical points of view, however, it is difficult to realize developed turbulence with sufficiently wide ISR. This is one of the main problems met in the study of intermittency in turbulence.

If the Reynolds number is only moderate, i.e., if a well developed ISR does not exist, one cannot clearly observe the power law relation (1), and no precise determination of $\zeta(q)$ is possible. Several years ago, Benzi and co-workers [5] empirically found that even when the Reynolds number is not really large, any two structure functions $S_q^u(r)$ and $S_p^u(r)$, q and p being arbitrary integers, show the mutual power law relation

$$S_q^u(r)\propto[S_p^u(r)]^{\zeta_p(q)}, \quad \zeta_p(q)=\zeta(q)/\zeta(p). \tag{2}$$

This mutual power law scaling [Eq. (2)], in terms of other structure functions instead of the scale r, seemed to hold over r ranges much larger than the region in which the original power law [Eq. (1)] is expected to hold. This finding is known as *extended self-similarity* (ESS) (second and fourth paper in [5]). After the papers by Benzi and co-workers, many studies were reported on this subject [6].

The main purpose of the present work is to study a scaling hypothesis in homogeneous, isotropic turbulence, and to show when it leads to ESS. This will provide a derivation of ESS together with insight into its range of validity. In Sec. II, in keeping with the conventional refined similarity hypothesis of Kolmogorov and Obukhov [7], we propose a scaling hypothesis for the velocity difference moments and for the coarse-grained energy dissipation rate fluctuations, which can be denoted as an extended refined similarity hypothesis (ERSH). The ERSH is characterized by the presence of a universal scaling function $g(x)$ with $x=r/L$, where L is the outer scale of the turbulent flow. Such a scaling function was already been introduced by Benzi and co-workers [5]; these authors restricted themselves to the ISR-VSR transition range, found ESS only down to 5η, and did not consider the apparent dependence of the scaling function on the moment order q in the viscous subrange (VSR). Here we will show how this $g(x)$-generalized scaling implies ESS for the structure functions of the velocity as well as of the energy dissipation rate fluctuations, and that it does so toward the outer scale while its validity toward small scales is deteriorated by intermittency and anomalous scaling exponents. The probability densities for the velocity differences and for the energy dissipation rate fluctuations, which are compatible with the ERSH, are obtained in Sec. III. The explicit form of the scaling function $g(x)$ which characterizes the ERSH toward the outer scale L is proposed in Sec. IV. Then Sec. V is concerned with ESS toward smaller, l-range or viscous subrange scales, l being the inner scale of the turbulent flow. The fluctuation spectrum, characterizing the probability densities, and its relation to the scaling exponents $\zeta(q)$, are dealt with in Sec. VI. Finally, a summary and remarks are given in Sec. VII.

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II. EXTENDED REFINED SIMILARITY HYPOTHESIS AND ESS

Let us first consider a fully developed turbulent flow, i.e., with a sufficiently large ISR. The Navier-Stokes dynamics is invariant under the transformation (see, e.g., Ref. [8])

$$r' = \lambda r, \quad u_{r'} = \lambda^{-z} u_r, \quad t' = \lambda^{1+z} t, \quad p_{r'} = \lambda^{-2z} p_r,$$
$$\epsilon_{r'} = \lambda^{-1-3z} \epsilon_r, \quad \nu' = \lambda^{1-z} \nu, \tag{3}$$

where p_r is the characteristic pressure change over the spatial scale r, ϵ_r is the local energy dissipation rate averaged over a region of linear extension r, ν is the kinematic viscosity, λ is an arbitrary positive constant, and z is an arbitrary exponent. Let L be the linear scale corresponding to the largest eddy motion, denoted as the *outer scale* of the system, and $U \equiv u_L$ be the typical velocity difference on this outer scale L, controlled by the external boundary conditions. Then L characterizes the crossover scale between the ISR and the stirring subrange, called the SSR. Note that under the above transformation [Eq. (3)] the Reynolds number $Re = UL/\nu$ does not change for any choice of λ and z. Assuming that ϵ_L, the energy dissipation rate averaged over the outer scale L, is constant in time and space, we introduce the scaling exponents z_r by

$$u_r \propto u_L \left(\frac{r}{L} \right)^{-z_r}. \tag{4}$$

Here z_r is a fluctuating variable [8] reflecting the u_r fluctuations, and $u_L \equiv (\epsilon_L L)^{1/3}$. The stochastic variable z_r is assumed to be stationary along the scale hierarchy in the sense that its statistical characteristics are independent of r in the ISR $l \ll r \ll L$. Here l is the inner scale; $l = a\eta$, where $a \approx 10$ (cf. first paper of Ref. [9]); and η is the Kolmogorov viscous length $\eta = (\nu^3/\epsilon)^{1/4}$. To express the assumed statistical r independence differently, $\langle (z_n - \langle z \rangle)(z_m - \langle z \rangle) \rangle$ is considered to decay very quickly with m differing from n. We then write z instead of z_r.

According to the invariance properties [Eqs. (3)], the dissipation rate scaling which corresponds to the velocity scaling [Eq. (4)] reads

$$\epsilon_r \propto \epsilon_L \left(\frac{r}{L} \right)^{-1-3z_r}. \tag{5}$$

A consequence of Eqs. (4) and (5) is the scaling relation

$$\frac{u_r}{u_L} \sim \left(\frac{\epsilon_r}{\epsilon_L} \right)^{1/3} \left(\frac{r}{L} \right)^{1/3}. \tag{6}$$

This is called the refined similarity hypothesis (RSH), introduced by Kolmogorov and Obukhov [7] (K62). The crucial assertion of the RSH is that the velocity fluctuations u_r and the energy dissipation rate fluctuations ϵ_r are related to each other by the same fluctuation exponent z_r. If we define the characteristic exponents $\zeta(q)$ and $\tau(q)$ in the ISR in terms of the respective structure functions for the velocity $S_q^u(r)$ [cf. Eq. (1)] and for the dissipation rate $S_q^\epsilon(r) \equiv \langle \epsilon_r^q \rangle$ by

$$S_q^u(r) \propto r^{\zeta(q)}, \tag{7a}$$

$$S_q^\epsilon(r) \propto r^{\tau(q)}, \tag{7b}$$

then Eqs. (4) and (5), via Eq. (6), lead to the bridging relation

$$\zeta(q) = \frac{q}{3} + \tau\left(\frac{q}{3} \right). \tag{8}$$

This implies, incidentally, $\zeta(0) = \tau(0) = 0$. The stationarity of ϵ_r in space implies that $\langle \epsilon_r \rangle = \epsilon_L$ is independent of r, which yields the relations $\tau(1) = 0$ and $\zeta(3) = 1$ [1,7]. If ϵ_r is assumed not to be fluctuating and also to be independent of r, Eq. (5) yields $z_r = -\frac{1}{3}$, which implies the classical Kolmogorov exponent $\zeta_{K41}(q) = q/3$. However, as is argued, the intermittency of the turbulent fluctuations changes the K41 exponent to $\zeta(q)$ and $\zeta(q) \neq \zeta_{K41}(q)$, the difference being $\tau(q/3)$. Much work was since devoted to this problem (see Refs. [2,3,8], etc.); Benzi *et al.* (1996) [5] also discussed the relation between the RSH and ESS.

As discussed above, if Re is large and turbulence is fully developed, power laws (7) can be observed in a wide range of r. However, in many cases Re is rather limited, and turbulence is not yet fully developed. Then no sufficiently large region of r exhibiting Eq. (7) might exist. This makes the analysis of numerical as well laboratory experiments difficult. Nevertheless, it turned out that the log-log plot of $S_q^u(r)$ vs $S_p^u(r)$ lies on a straight line in a range much wider than the log-log plot of $S_q^u(r)$ vs r itself. This empirical fact was first pointed out by Benzi and co-workers [5], and is called extended self-similarity. It is quite an interesting finding. One may analogously consider $\langle \epsilon_r^q \rangle$ vs $\langle \epsilon_r^p \rangle$, and conjecture that ESS also holds for these r-averaged energy dissipation rate structure functions. Our paper addresses the question of how to explain the ESS originally found by Benzi and co-workers, although only down to 5η, to both the VSR and SSR by introducing a scaling hypothesis. We then extend this to the possibility of an ESS for the ϵ_r structure functions as well.

Let us first note that scalings (4) and (5) are a result of a kind of dimensional analysis despite the presence of fluctuations. Now the units of u_r and ϵ_r do not change by multiplying them by dimensionless factors. In the present paper, in keeping with this observation, instead of the scaling formulas (4) and (5) we propose the following extended scaling, expected to hold in larger r ranges or even for all relevant r:

$$u_r \propto u_L \left[\frac{r}{L} g\left(\frac{r}{L} \right) \right]^{-z_r}, \tag{9}$$

$$\epsilon_r \propto \epsilon_L \left[\frac{r}{L} g\left(\frac{r}{L} \right) \right]^{-1-3z_r}. \tag{10}$$

Here $g(x)$ is taken to be a dimensionless, unique, monotonically decreasing, universal function of $x = r/L$, which has to satisfy $g(x) = 1$ for $l/L \ll x \ll 1$. Furthermore, since no velocity correlations exist for distances $r \gg L$, we require that u_r and ϵ_r then become independent of r, implying $g(x)$

$=\text{const}\times x^{-1}$ for $1\ll x$. In the opposite case of $x\ll l/L$ the function $g(x)$ has to lead as $S_q^u(r)\propto x^q$ in order to reflect the regular behavior of the structure function for $r\to 0$.

Because of these features of the scaling function $g(x)$, scalings (9) and (10) reduce to Eqs. (4) and (5), if the ISR is sufficiently large and r is chosen well between l and L. In the same manner as one derives the refined similarity [Eq. (6)] from the u_r and ϵ_r fluctuation representations in terms of z, from the extended scaling formulas (9) and (10) one arrives at the following ERSH:

$$\frac{u_r}{u_L}\propto\left(\frac{\epsilon_r}{\epsilon_L}\right)^{1/3}\left[\frac{r}{L}g\left(\frac{r}{L}\right)\right]^{1/3}. \tag{11}$$

This is the fundamental relation, in addition to Eqs. (9) and (10), to derive ESS in the sense of Benzi and co-workers [5].

Let us introduce the scales r_n by the implicit definition

$$\frac{r_n}{L}g\left(\frac{r_n}{L}\right)\equiv b^{-n},\quad n=0,1,2,\ldots,N. \tag{12}$$

Here b, with $b>1$, is an arbitrary but fixed constant, r_0 being the solution of $(r_0/L)g(r_0/L)=1$, and N is chosen such that r_N is in the crossover range between the VSR and the ISR. Note that g depends on the r_n; in particular these are not a simple geometrically decreasing sequence of scales, unless $g=1$, as is the case in the ISR.

Incidentally, the extended scaling formulas (9) and (10), together with the statistical r independence of z, allow one to derive, for the velocity and the dissipation rate ratios, the following freqently used expressions:

$$\frac{u_{r_{n+1}}}{u_{r_n}}=b^z,\quad \frac{\epsilon_{r_{n+1}}}{\epsilon_{r_n}}=b^{1+3z}. \tag{13}$$

The r-invariant scaling ratios are usually considered in the ISR only. Now they are extended to larger r ranges by including the crossover function $g(x)$. The statistical stationarity of the ratios is equivalent to the statistical stationarity of the fluctuation variables z_r, i.e., their statistical r independence. We keep this now common assumption, and substitute z_r for z in Eqs. (9) and (10).

Inserting Eq. (12) into Eqs. (9) and (10) yields

$$u_{r_n}=u_Lb^{nz},\quad \epsilon_{r_n}=\epsilon_Lb^{n(1+3z)},\quad n=0,1,2,\ldots,N. \tag{14}$$

The structure functions $S_q^u(r)$ and $S_q^\epsilon(r)$ at the scales $r=r_n$ are thus evaluated as

$$S_q^u(r_n)\propto u_L^q\langle b^{nqz}\rangle\equiv u_L^q b^{-n\zeta(q)}, \tag{15}$$

$$S_q^\epsilon(r_n)\propto\epsilon_L^q b^{qn}\langle b^{3qnz}\rangle=\epsilon_L^q b^{qn}b^{-n\zeta(3q)}\equiv\epsilon_L^q b^{-n\tau(q)}, \tag{16}$$

where we have defined the characteristic exponent function

$$\zeta(q)=-\frac{1}{n}\log_b\langle b^{nqz}\rangle. \tag{17}$$

Here we assume that $\zeta(q)$ is independent of n for $1<n<N$. We shall show momentarily that this definition of $\zeta(q)$ coincides with the one given in Eq. (1). The average $\langle\cdots\rangle$ is taken over the randomness of z. Furthermore, $\tau(q)$ was defined through $q-\zeta(3q)=-\tau(q)$ in Eq. (16), which is the same as in Eq. (8).

From the explicit representation [Eq. (17)] for the scaling exponents, one can draw the general conclusion that $\zeta(q)$ is convex curved downward. It therefore obeys the following inequalities [10]:

$$\frac{d^2\zeta(q)}{dq^2}\leqslant 0, \tag{18a}$$

$$\frac{d}{dq}\left[\frac{\zeta(q)}{q}\right]\leqslant 0. \tag{18b}$$

These inequalities play an important role when making probabilistic models of intermittency; for examples, see Ref. [11].

The convexity downward follows if one applies Schwarz's inequality: $\langle b^{nz(q_1+q_2)}\rangle\leqslant[\langle b^{nz2q_1}\rangle\langle b^{nz2q_2}\rangle]^{1/2}$. Inserting this into representation (17) yields

$$\zeta\left(\frac{p_1+p_2}{2}\right)\geqslant\frac{1}{2}(\zeta(p_1)+\zeta(p_2)). \tag{17$'$}$$

Convexity downward shows that the slope $\zeta'(q)$ can only decrease, i.e., $\zeta''\leqslant 0$, or Eq. (18a). It also shows that for any interval the mean slope is larger than the slope on the right end of that interval. Applying this statement to the interval $[0,q]$ implies that $\zeta(q)/q\geqslant\zeta'(q)$, because $\zeta(0)=0$ or, equivalently, that $q\zeta'-\zeta\leqslant 0$, which after dividing by q^2 is the second inequality [Eq. (18b)].

Eliminating b from Eqs. (15) and (16) by substituting Eq. (12), we obtain

$$S_q^u(r)\propto u_L^q\left[\frac{r}{L}g\left(\frac{r}{L}\right)\right]^{\zeta(q)},\quad S_q^\epsilon(r)\propto\epsilon_L^q\left[\frac{r}{L}g\left(\frac{r}{L}\right)\right]^{\tau(q)}. \tag{19}$$

We require that if r is much smaller than L, by noting that $g(0)=1$, Eqs. (19) should coincide with Eqs. (7). This proves that $\zeta(q)$ and $\tau(q)$ introduced by definitions (15) and (16) are the same as those conventionally defined in Eqs. (7).

We can now draw conclusions from the generalized equations (19): If r/L is not within the inertial subrange $l/L\ll r/L\ll 1$, the log-log plot of structure functions (19) vs r do *not* exhibit a linear relation because $g(r/L)$, outside of the ISR or in the lower and upper transition ranges beyond the ISR, depends on r/L; therefore Eq. (19) does not coincide with (7). Nevertheless, if we plot $S_q^u(r)$ and $S_q^\epsilon(r)$ as functions of $S_p^u(r)$ and $S_p^\epsilon(r)$, respectively, by eliminating $xg(x)$ with $x=r/L$ we obtain

$$S_q^u(r)\propto u_L^{q-p\zeta_p(q)}[S_p^u(r)]^{\zeta_p(q)},$$

$$S_q^\epsilon(r)\propto\epsilon_L^{q-p\tau_p(q)}[S_p^\epsilon(r)]^{\tau_p(q)}. \tag{20}$$

Again, the previous definition given in Eq. (2) has been used:

$$\zeta_p(q)=\frac{\zeta(q)}{\zeta(p)}, \quad \tau_p(q)=\frac{\tau(q)}{\tau(p)}. \tag{21}$$

One should note that the equality $\tau(1)=0$ implies that $\tau_1(q)$ cannot be defined. This stems from the fact that S_1^{ϵ} is independent of r, and therefore cannot be used as a substitute for r. Furthermore, again eliminating $(r/L)g(r/L)$ in Eqs. (19) but inserting the ϵ-structure function formula into the u-structure function formula, or vice versa, we obtain

$$S_q^u(r)\propto u_L^q \epsilon_L^{-p\zeta(q)/\tau(p)}[S_p^{\epsilon}(r)]^{\zeta(q)/\tau(p)},$$

$$S_q^{\epsilon}(r)\propto \epsilon_L^q u_L^{-p\tau(q)/\zeta(p)}[S_p^u(r)]^{\tau(q)/\zeta(p)}. \tag{22}$$

The second equations of Eqs. (20) and (22) represent an ESS first proposed in the present paper. Last but not least, making use of the compensated expressions for the structure functions [9], we predict that scaling relations, using $\zeta(3)=1$, can be concluded from the Howard–von Kármán–Kolmogorov structure equation only if the longitudinal velocity difference itself is taken and not only its magnitude u_r (as we do here). The presented formulas can well be extended to include the possibility $\zeta(3)\neq 1$. However, note that ESS is *not* derived here for odd order structure functions which include the signs of the velocity differences. See also the second of Refs. [9]. Furthermore, we obtain

$$\frac{S_q^u(r)}{[S_3^u(r)]^{q/3}}\propto u_L^{-p\tau(q/3)/\zeta(p)}[S_p^u(r)]^{\tau(q/3)/\zeta(p)}$$

$$\propto \epsilon_L^{-p\tau_p(q/3)}[S_p^{\epsilon}(r)]^{\tau_p(q/3)}. \tag{23}$$

The above considerations show that even if the structure functions have no pure power law dependence on r, as is the case if turbulence is not sufficiently fully developed, and crossovers to the SSR or the VSR are significant, the above power law relations *between* the structure functions themselves hold, provided a proper $g(x)$ can be found. The first equation of Eqs. (20) is identical to the conventional ESS, found empirically [5,6]. The second equation of Eqs. (20), and both Eqs. (22), are types of ESS first derived in the above discussion. Also Eq. (23), a *compensated* bridging expression, is given here first. The important advantage of Eq. (23) is, that the right-hand side has an exponent which measures the intermittency corrections $\tau(q/3)$ only. If one plots this, the scale can be much more stretched, and can even be nonlogarithmic, thus enhancing the visibility of the intermittency corrections. This eases their identification; see Ref. [9].

III. PROBABILITY DENSITIES FOR u_r AND ϵ_r

The probability densities $P_r(u)$ for u_r and $P_r(\epsilon)$ for ϵ_r can be evaluated, remembering expression (4) and its present generalization $u_r\propto u_L(xg(x))^{-z_r}$ with $x=r/L$, by means of the probability densities Q for the exponent fluctuations. These distributions Q depend, for a given r/L, on the stochastic variables z_r, and thus read $Q_r(z_r)$. The relation holds for $P_r(u)|du|=Q_r(z_r)|dz_r|$. If the random exponents z_r are statistically independent of r, the probability densities $Q_n(z)$ at the discrete series r_n of scales [cf. Eq. (12)] take the asymptotic form [10]

$$Q_n(z)\propto \sqrt{n}\, b^{-S(z)n} \tag{24}$$

for large n or, equivalently, $r_n \ll L$. This result was obtained in an analysis denoted as *large deviation theory* [10]. The function $S(z)$ is called the fluctuation spectrum of z. It is independent of n, and is expected to be a universal function, which characterizes the fluctuations of the magnitude of the velocity differences u_r and of the r-averaged energy dissipation rates ϵ_r in turbulence via Eqs. (4) and (5). An ergodicity ansatz for the z fluctuations implies that $Q_n(z)$ have a single peak structure around the ensemble average $\langle z\rangle$ of z, the value which minimizes $S(z)$. Furthermore, assuming that $S(z)$ has no inflection point, $S(z)$ turns out to be a concave function [10]:

$$S''(z)>0. \tag{25}$$

Knowing $S(z)$ and thus the distributions $Q_n(z)$, one finds $P_{r_n}(u)=|du/dz|^{-1}Q_n(z(u))$ and $P_{r_n}(\epsilon)=|d\epsilon/dz|^{-1}Q_n(z(u))$. Eliminating z and n in these expressions with the use of $u=u_L b^{nz}$ and $\epsilon=\epsilon_L b^{n(1+3z)}$, as given in Eqs. (14), and finally replacing r_n by r, we arrive at

$$P_r(u)\propto u^{-1}\left[\frac{r}{L}g\left(\frac{r}{L}\right)\right]^{S(\ln(u/u_L)/\ln[(r/L)g(r/L)]^{-1})} \times \frac{1}{\sqrt{\ln\left[\frac{r}{L}g\left(\frac{r}{L}\right)\right]^{-1}}}, \tag{26}$$

$$P_r(\epsilon)\propto \epsilon^{-1}\left[\frac{r}{L}g\left(\frac{r}{L}\right)\right]^{S((-1+\ln(\epsilon/\epsilon_L)/\ln[r(/Lg)(r/L)]^{-1}/3))} \times \frac{1}{\sqrt{\ln\left[\frac{r}{L}g\left(\frac{r}{L}\right)\right]^{-1}}}. \tag{27}$$

The structure functions can be written in terms of the probability densities (24) and $P_{r_n}(u)\propto u^{-1}Q_n(z(u))$ as

$$S_q^u(r_n)=\int_0^{\infty}u^q P_{r_n}(u)du\propto u_L^q\int_{-\infty}^{\infty}b^{n[qz-S(z)]}dz. \tag{28a}$$

Analogously with the corresponding ϵ_r distributions, one obtains

$$S_q^{\epsilon}(r_n)=\int_0^{\infty}\epsilon^q P_{r_n}(\epsilon)d\epsilon\propto \epsilon_L^q\int_{-\infty}^{\infty}b^{n[(1+3z)q-S(z)]}dz. \tag{28b}$$

Applying the method of steepest descent for large n and comparing with Eqs. (15) and (16) immediately yields

$$\zeta(q)=\min_z[S(z)-qz], \tag{29}$$

and also formula (8), relating $\tau(q)$ with $\zeta(q)$. Equation (29) can be solved conveniently if the minimum is in the interior of the interval on which $S(z)$ is defined. We define the function $z(q)$ as the solution of

$$S'(z(q))=q. \tag{30}$$

Then

$$\zeta(q)=S(z(q))-qz(q). \tag{31}$$

This relation is useful in determining the fluctuation spectrum $S(z)$ from experiment. If the sets $(q,\zeta(q))$ are known from experiment by evaluating the structure function exponents, one can obtain the relation $(z,S(z))$ by numerically carrying out the Legendre transform (31).

IV. ISR-SSR CROSSOVER

Batchelor [12] first introduced the empirical interpolation or crossover function for the second order velocity structure function bridging the statistics of the longitudinal velocity components in the inertial (ISR) and viscous (VSR) subranges:

$$S_2^u(r)=\frac{\epsilon_L}{15\nu}\frac{r^2}{\left[1+\left(\frac{r}{l}\right)^2\right]^{[2-\zeta(2)]/2}}. \tag{32}$$

Again we have $l=a\eta$, with the Kolmogorov microscale η. From Eq. (32) one has $S_2^u(r)=(\epsilon_L/15\nu)r^2$ in the VSR ($r \ll l$) and $(a^2/15)(\epsilon_L\eta)^{2/3}(r/l)^{\zeta(2)}$ in the ISR ($r\gg l$). The latter expression more conventionally is written as $b_\parallel(\epsilon_L r)^{2/3}(r/L)^{\tau(2/3)}$.

Then, extending this empirical formula (32) to comprise the transition to the SSR as well, $r\geqslant L$, Lohse and Müller-Groeling [13] proposed the more general interpolation formula

$$S_2^u(r)=\frac{a^2}{15}(\epsilon_L\eta)^{2/3}\frac{\left(\frac{r}{l}\right)^2}{\left[1+\left(\frac{r}{l}\right)^2\right]^{[2-\zeta(2)]/2}\left[1+\left(\frac{r}{L}\right)^2\right]^{\zeta(2)/2}}. \tag{33}$$

This formula was successfully applied to data analysis by Lohse and Müller-Groeling and by Grossmann and coworkers [9]. Here let us slightly generalize formula (33) by writing

$$S_2^u(r)=\frac{a^2}{15}(\epsilon_L\eta)^{2/3}\frac{\left(\frac{r}{l}\right)^2}{\left[1+\left(\frac{r}{l}\right)^{\kappa_l}\right]^{[2-\zeta(2)]/\kappa_l}\left[1+\left(\frac{r}{L}\right)^{\kappa_L}\right]^{\zeta(2)/\kappa_L}}. \tag{34}$$

Two additional arbitrary constants κ_l and κ_L have been introduced, describing the respective widths of the crossover regions. Taking $\kappa_l=\kappa_L=2$ corresponds to the original interpolation [Eq. (33)]. One easily finds that Eq. (34) yields the asymptotics $S_2^u(r)\propto r^2$ for $r\ll l$ and $S_2^u(r)\propto r^{\zeta(2)}$ for $l\ll r \ll L$, irrespective of the value of κ_l, and $S_2^u=$const, if $L\ll r$, again irrespective of κ_L. We now discuss first the region $l \ll r$. Then formula (34) reduces to

$$S_2^u(r)=\frac{a^{2-\zeta(2)}}{15}\left(\frac{L}{\eta}\right)^{\zeta(2)-(2/3)}u_L^2\left[\frac{r}{L}\cdot\frac{1}{\left[1+\left(\frac{r}{L}\right)^{\kappa_L}\right]^{1/\kappa_L}}\right]^{\zeta(2)}. \tag{35}$$

Comparing this interpolation formula (35) with $S_2^u(r) \propto u_L^2[(r/L)g(r/L)]^{\zeta(2)}$ from Eq. (19) in the particular case $q=2$, for the crossover function we find

$$g(x)=\frac{1}{(1+x^{\kappa_L})^{1/\kappa_L}} \tag{36}$$

without any constant factor.

The outer scale L characterises the crossover range between the inertial and the stirring subranges, the ISR and the SSR. The fluctuation statistics changes within this crossover region. The width of this crossover range depends on the moment order q. We shall estimate this now. Let us rewrite the first equation of Eqs. (19) by making use of the explicit form of the scaling function $g(x)$ according to Eq. (36) as

$$S_q^u(r)\propto u_L^q f_q(r), \quad f_q(r)=\left[1+\left(\frac{L}{r}\right)^{\kappa_L}\right]^{-\zeta(q)/\kappa_L}. \tag{37}$$

Note that $l\ll r$ is considered. $S_q^u\propto r^{\zeta(q)}$ for $l\ll r\ll L$ and $S_q^u \propto$const for $L\ll r$ are recovered from Eq. (37).

The ISR-SSR crossover takes place at $r\approx L$, independent of q by construction of $g(x)$. The crossover range width is written as $2\delta L_q$. The lower scale $L_q\equiv L-\delta L_q$, where the crossover starts, is defined as that scale r for which $S_q^u(r)$, viz. $f_q(r)$, has just half the magnitude it has at the crossover center L itself:

$$f_q(r=L_q)=\frac{1}{2}f_q(r=L). \tag{38}$$

This is a possible and reasonable definition, because $f_q(r)$ is an increasing function of r for q, for which $\zeta(q)>0$. On the other hand, for q with $\zeta(q)<0$, $f_q(r)$ is a decreasing function of r. We then define L_q by

$$f_q(r=L_q)=2f_q(r=L). \tag{39}$$

Using these definitions and inserting $f_q(r)$ from Eq. (37) we obtain the crossover scale L_q as

$$L_q = L[2^{1+\kappa_L/|\zeta(q)|} - 1]^{-1/\kappa_L}, \quad \delta L_q = L - L_q. \tag{40}$$

Thus, indeed, the width δL_q of the crossover range, which is positive, depends on the moment order q. Particularly, noting $\zeta(0)=0$, $\zeta(3)=1$, and $|\zeta(\pm\infty)|=\infty$, we find $L_0=0$, $L_3 = L/\sqrt{7} \approx 0.38L$, and $L_{\pm\infty}=L$ for $\kappa_L=2$, corresponding to $\delta L_0 = L$, $\delta L_3 = 0.62L$, and $\delta L_\infty = 0$. Thus, if we look at the fluctuations through the ''filtering'' $q=0$, there exists no ISR. The general observation is that the crossover range shrinks with increasing moment order q.

V. ESS TOWARD SMALLER SCALES, THE VSR?

We have derived that ESS is valid toward and well into the stirring subrange SSR by generalizing the scaling variable from r/L to $(r/L)g(r/L)$, i.e., by introducing a crossover function $g(r/L)$. This ESS confirmation is in good agreement with the results obtained in the second of Refs. [9], where the extension of the ESS toward larger scales was first noted. The original idea of the ESS was more directed toward smaller scales [5]. Empirically there was quite a bit of support for this hypothesis. Therefore the observations in Ref. [9] that it is *large* scales instead of small ones toward which ESS holds, came with some surprise. We now discuss the question of ESS toward the VSR scales by again using a crossover scaling variable $(r/l)g(r/l)$, now in the l crossover range.

Consider the generalized interpolation formula (34). Now take eddy scales r in the inertial and viscous subranges, but well below the outer scale L. With the abbreviation $y=r/l$, from Eq. (34) we have

$$S_2^u(y) \propto \frac{y^2}{(1+y^{\kappa_l})^{[2-\zeta(2)]/\kappa_l}}, \tag{41}$$

leading to $\propto y^2$ for $y=r/l \ll 1$ and to the ISR scaling $\propto y^{\zeta(2)}$ for $1 \ll y = r/l$ but still $y \ll L/l$. In order to compare with the pure power law [Eq. (1)] or with the generalized power law [Eq. (19)], the latter having the crossover function g included, reading here as

$$S_2^u(y) \propto \left[\frac{r}{l} g\left(\frac{r}{l}\right)\right]^{\zeta(2)}, \tag{42}$$

we rearrange expression (41) into

$$S_2^u(y) \propto \left[\left(\frac{y^{\kappa_l}}{1+y^{\kappa_l}}\right)^{2/(\kappa_l \zeta(2))} (1+y^{\kappa_l})^{1/\kappa_l}\right]^{\zeta(2)} \equiv [yg(y)]^{\zeta(2)}. \tag{43}$$

In contrast to the ISR-SSR crossover function g given as Eq. (36), $g(y)$ here depends on the anomalous scaling exponent $\zeta(2)$:

$$g(y) = (1+y^{-\kappa_l})^{-(1/\kappa_l)([2/\zeta(2)]-1)} = \begin{cases} y^{[2/\zeta(2)]-1}, & y \ll 1, \\ 1, & y \gg 1. \end{cases} \tag{44}$$

This crossover function shows, in particular, the correct scaling in the VSR, $S_2^u \propto (yg(y))^{\zeta(2)} = y^2$, and in the ISR one correctly finds $S_2^u \propto (yg(y))^{\zeta(2)} = y^{\zeta(2)}$. If we write the general scaling with this g,

$$S_q^u \propto \left[\frac{r}{l} g\left(\frac{r}{l}\right)\right]^{\zeta(q)}, \tag{45}$$

it depends on $\zeta(2)$, i.e., on the intermittency correction itself. For $y \ll 1$ we have $S_q^u \propto y^{2\zeta(q)/\zeta(2)}$, which does not have the regular power law behavior $S_q^u \propto y^q$ for $y \propto r \to 0$. We have to conclude that ESS *cannot* be extended toward the VSR, even if a crossover function is introduced. This is in perfect agreement with the results found in Ref. [9], second paper.

In the case of K41 turbulence, i.e., without intermittency influences, $g(y)$ from Eq. (44) with $\zeta(2)=2/3$ simplifies to

$$g_{K41}(y) = (1+y^{-\kappa_l})^{-2/\kappa_l} = \begin{cases} y^2, & y \ll 1, \\ 1, & y \gg 1. \end{cases} \tag{46}$$

Now, in K41 turbulence, for which we have $\zeta(q)=q/3$, one *can* derive a generalized ESS, using $g(x)$ from Eq. (46):

$$S_q^u \propto (yg(y))^{\zeta(q)} = \begin{cases} y^q, & y \ll 1, \\ y^{q/3}, & y \gg 1. \end{cases} \tag{47}$$

All ESS formulas (20)–(23) are valid for K41 turbulence (although one does not really need them in this case) because the factors $\frac{1}{3}$ cancel in $\zeta_p(q)$. We conclude that it is the *deviation* of $\zeta(2)$ from the K41 value 2/3 which prohibits introducing a universal crossover function $g(y)$. Since $\zeta(2) - \zeta_{K41}(2) = \tau(2/3)$ is very small, ≈ 0.03, ESS is *nearly* valid for intermittent turbulence; *nearly* means ''order of intermittency corrections.'' In particular, in log-log plots over typically rather restricted r ranges in the VSR of about a one-fifth of a decade, this deviation will hardly be visible for a not too large value of q. Therefore ESS at least approximately also holds toward the VSR in intermittent turbulence, although strictly speaking it is not valid. However, note that in plots with compensated structure functions, which are sensitive to intermittency corrections, one will find that ESS does not hold because here these corrections are the leading terms. All this is in good agreement with the results of Benzi and co-workers [5] as well as of Grossmann, Lohse, and Reeh [9]. In particular this only approximate validity of ESS, deteriorated by the anomalous scaling deviations, allows one to understand the empirical finding [5] that ESS can be seen only for $r \geq 5\eta$.

VI. FLUCTUATION SPECTRUM

We now determine the fluctuation spectrum $S(z)$ of the velocity difference fluctuations $u \propto (r/L)^{-z}$. To do this we consider stochastic models as, e.g., the log-normal or log-Poisson models. The log-normal model [2,7,14] yields a parabolic form of $S(z)$, and is given in terms of one parameter only, the intermittency exponent μ (≈ 0.2), as

$$\tau(q) = \frac{\mu}{2} q(1-q),$$

implying

$$S(z) = \frac{9}{2\mu}\left(z + \frac{1}{3} + \frac{\mu}{6}\right)^2. \tag{48}$$

$S(z)$ here was obtained from $\tau(q)$ via $\zeta(q)$ and the Legendre transform algorithm, described above in Sec. III.

Very recently, Watanabe and one of the authors [11] discussed the q dependence of the moment exponents $\zeta(q)$ as well as the spectrum $S(z)$ of the She-Leveque model, which is a log-Poisson model [15]. A comparison with the β model [16] was made, and the differences between these two models were elucidated. In Ref. [11] it was shown that for large q (with $q>0$) the exponent function $\zeta(q)$ and the corresponding [cf. Eqs. (30) and (31)] fluctuation spectrum $S(z)$ can be described by the expansions

$$\tau(q) = -\gamma q + d_0 - Ce^{-q/q_*}, \tag{49}$$

$$S(z) = d_0 - 3q^*\left(z + \frac{1-\gamma}{3}\right)\ln\left[\frac{3q^*}{eC}\left(-z - \frac{1-\gamma}{3}\right)\right]. \tag{50}$$

Particularly, employing She-Leveque's argument [15], we obtain

$$d_0 = C = 2, \quad \gamma = \frac{2}{3}, \quad e^{-1/q^*} = 1 - \frac{\gamma}{d_0} = \frac{2}{3}, \tag{51}$$

which yields

$$\tau(q) = -\frac{2}{3}q + 2\left[1 - \left(\frac{2}{3}\right)^q\right], \tag{52a}$$

$$\zeta(q) = \frac{q}{3} + \tau\left(\frac{q}{3}\right) = \frac{q}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{q/3}\right]. \tag{52b}$$

From Eq. (50) with Eqs. (52a) and (52b), one finds

$$S(z) = 2 - \frac{3}{\ln\frac{3}{2}}\left(z + \frac{1}{9}\right)\ln\left[\frac{3}{2e\ln\frac{3}{2}}\left(-z - \frac{1}{9}\right)\right]. \tag{53}$$

This expression is valid in a region where $z \leqslant -1/9$, more generally $z \leqslant -(1-\gamma)/3$. In K41 turbulence we have $z = -1/3$, which *is* less than $-1/9$. Equation (53) is the explicit formula for the fluctuation spectrum $S(z)$ valid in She-Leveque's log-Poisson model [17]. $S(z)$ can be described as linear in z with logarithmic corrections.

VII. SUMMARY AND REMARKS

In the present paper, we first introduced a scaling variable $xg(x)$ to include crossovers between various subranges of scaling behavior for the magnitude of longitudinal velocity differences in turbulence. This turned out to lead to extended self-similarity (ESS) in a quite natural way. We showed that the refined similarity hypothesis (RSH) (see Ref. [7]) can be generalized by considering a scaling function $g(r/L)$. The conventional refined similarity hypothesis is incompatible with the ESS, but the present similarity hypothesis holds in a much larger region which includes the stirring subrange (SSR). The fact that ESS is much more valid toward large scales can also be found in two-dimensional magneto hydrodynamics simulations [18]. Although extension towards VSR, strictly speaking, is not valid, it nevertheless holds at least approximately, the deviations being of the order of the intermittency corrections. With the scaling function $g(r/L)$ the extended refined similarity hypothesis leads to the experimentally and numerically observed ESS.

We also found that our proposed scaling hypothesis predicts ESS for energy dissipation rate fluctuations as well as for the bridging relations between the structure functions of the velocity differences and those of the energy dissipation rate. Also, the considerable advantage of using compensated structure functions was discussed. Furthermore, making use of an extended form of the Batchelor parametrization, we obtained explicit formulas for the scaling function $g(x)$, cf. Eqs. (36) and (46). Of course, these can be put together analogously, as indicated in Eq. (34). The exponent function $\zeta(q)$ and the eddy fluctuation spectrum $S(z)$ were illustrated for the log-normal and log-Poisson models. The present theoretical results still have to be compared with measured data.

Let us finally comment on the probability density [Eq. (26)]. For the scale r in the ISR of developed turbulence, the velocity difference is scaled as $u_r \sim r^h$, h being equal to the exponent $(-z_r)$ in Eq. (4). The probability density to observe a local scaling exponent with the value h is proportional to $r^{3-D(h)}$ [8], where $D(h)$ is the fractal dimension, and is related to $S(z)$ in Eq. (24) via

$$S(z) = 3 - D\left(\frac{1-z}{3}\right) \tag{54}$$

[3]. Benzi *et al.* (1996) [5] and Meneveau [6] proposed a multifractal theory of the probability density for the velocity differences similar to Eq. (26) to find compatibility with ESS. From the definition of the fluctuation spectrum $S(z)$ [Eq. (24)], we have $S(z) \geqslant 0$ for the possible values of z, which means $D(h) \leqslant 3$. If $S(z) > 3$ for some z, $D(h)$ takes negative values. This implies that although the *dimension* is conventionally positive, one should be aware that the fractal dimension can take negative values.

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[1] A. N. Kolmogorov, Dokl. Akad. Nauk. SSSR **30**, 9 (1941).
[2] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975).
[3] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[4] T. Bohr, M. H. Jensen, G. Paladin, and A. Vulpiani, *Dynamical Systems Approach to Turbulence* (Cambridge University Press, Cambridge, 1998).
[5] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, and S. Succi, Phys. Rev. E **48**, R29 (1993); R. Benzi, S. Ciliberto, C. Baudet, and G. R. Chavarria, Physica D **80**, 385 (1995); M. Briscolini, P. Santangelo, S. Succi, and R. Benzi, Phys. Rev. E **50**, R1745 (1994); R. Benzi, L. Biferale, S. Ciliberto, M.V. Struglia, and R. Tripiccione, Physica D **96**, 162 (1996).
[6] V. Carbone, P. Veltri, and R. Bruno, Phys. Rev. Lett. **75**, 3110 (1995); A. Arneodo *et al.*, Europhys. Lett. **34**, 411 (1996); G. S. Lewis and H. L. Swinney, Phys. Rev. E **59**, 5457 (1999); R. Benzi, L. Biferale, S. Ciliberto, M. V. Struglia, and R. Tripiccione, *ibid.* **53**, R3025 (1996); C. Meneveau, *ibid.* **54**, 3657 (1996).
[7] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962); A. M. Obukhov, *ibid.* **13**, 77 (1962).
[8] G. Parisi and U. Frisch, in *Turbulence and Predictability in Geophysical Fluid Dynamics,* Proceedings of the International School of Physics ''E. Fermi,'' 1983, Varenna, Italy, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 84.
[9] H. Effinger and S. Grossmann, Z. Phys. B: Condens. Matter **66**, 289 (1987); S. Grossmann, D. Lohse, and A. Reeh, Phys. Rev. E **56**, 5473 (1997); S. Grossmann, D. Lohse, and A. Reeh, Phys. Fluids **9**, 3817 (1997); S. Grossmann, D. Lohse, and A. Reeh, Phys. Rev. E **61**, 5195 (2000).
[10] H. Fujisaka and M. Inoue, Prog. Theor. Phys. **77**, 1334 (1987).
[11] T. Watanabe and H. Fujisaka, J. Phys. Soc. Jpn. **69**, 1672 (2000).
[12] G. K. Batchelor, Proc. Cambridge Philos. Soc. **47**, 359 (1951); H. Effinger and S. Grossmann, Z. Phys. B: Condens. Matter **66**, 289 (1987).
[13] D. Lohse and A. Müller-Groeling, Phys. Rev. Lett. **74**, 1747 (1995); D. Lohse and A. Müller-Groeling, Phys. Rev. E **54**, 395 (1996).
[14] A. S. Gurvich and A. M. Yaglom, Phys. Fluids **10**, S59 (1967).
[15] Z.-S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994); B. Dubrulle, *ibid.* **73**, 959 (1994); Z.-S. She and E. C. Waymire, *ibid.* **74**, 262 (1995).
[16] U. Frisch, P.-L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).
[17] T. Watanabe, Y. Nakayama, and H. Fujisaka, Phys. Rev. E **61**, R1024 (2000).
[18] H. Politano, A. Pouquet, and V. Carbone, Europhys. Lett. **43**, 516 (1998).